# Conditional Performance Error Covariance Analyses for Commercial Titan Launch Vehicles

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**An important part of space launch vehicle mission planning is the analysis of performance dispersions at various trajectory events such as park orbit injection and payload separation. Performance dispersions produce position, velocity, and time errors at these events. The variances and covariances of the performance errors are usually specified by a 7×7 space-time error covariance matrix. At a predetermined mission time, the position and velocity errors are viewed as being conditional on the predefined time, and there exists a need to compute a conditional 6×6 space covariance matrix. A numerical approach is usually used to compute the conditional matrix. In this paper, a comprehensive theoretical method is developed to transform a 7×7 (or higher order) space-time covariance matrix into a 6×6 space covariance matrix for a Commercial Titan launch vehicle. The method is based on the statistical theory of conditional multivariate normal distributions. Using chi-square ($\chi^2$)-based based equality of variance hypothesis tests, the theoretical matrix is compared to the numerically computed matrix. The hypothesis test results establish that there are no significant differences between the theoretically and numerically computed 6×6 conditional space covariance matrices.**

## Introduction

To compute the spacecraft fuel requirements needed to correct for injection errors, the performance dispersions of a launch vehicle must be determined accurately. In addition to the accumulated performance errors resulting from uncertainties in thrust, specific impulse, and propellant weight, etc., errors are imparted to the vehicle due to main engine shutdown time granularity and the tailoff impulse uncertainties. A velocity trim burn is performed after the main engine shutdown by firing attitude control system engines. Although the velocity trim burn reduces the performance errors, it results in a wide variation in the park orbit injection time. Therefore, the performance dispersions of the booster launch vehicle result in position, velocity, and timing errors at park orbit injection. These errors are carried over to the payload deployment event and produce errors in the deployment position, velocity, and time.

The variances and covariances of the performance errors are specified by a 7×7 space-time covariance matrix. To realistically compute the orbital element errors using numerical techniques such as Monte Carlo simulations, it is necessary to reduce the 7×7 space-time covariance matrix into a conditional 6×6 covariance matrix of only the state vector. This reduction is typically done by using numerical techniques. The purpose of this paper is to bring light upon the statistical and physical theory, and to validate the numerical computation. This is achieved by developing a theoretical conditional state covariance matrix using the statistical theory of conditional multivariate normal distributions and performing variance equality hypothesis tests between the theoretically and numerically computed conditional state covariance matrices. This requires a basic understanding of the launch vehicle, the mission sequence, statistical performance dispersion modeling, sensitivity matrix theory, and the use of variance equality hypothesis testing.

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## Vehicle and Mission Descriptions

A Commercial Titan is a three-stage space launch vehicle: stages 0, I, and II. Stage 0 consists of two solid rocket motors strapped to opposite sides of stages I and II. The Commercial Titan booster is similar to the Titan IIIC configuration.[1] For a typical dual payload mission, the spacecraft is injected into a 100×375-n.mi. nominal park orbit at the end of the velocity trim burn. Following a continuous roll maneuver for thermal control during coast, a typical payload separation includes predicting the nodal crossing time to correct for booster performance dispersions and spinning up the payload. Each payload is normally separated near an ascending or descending node, which is followed by the payload performing several perigee and apogee kick motor burns in order to inject the spacecraft into a geosynchronous orbit.

The spacecraft fuel requirement needed to correct for injection errors depends on the booster performance dispersions. The errors at park orbit injection include the position, velocity, and time errors. The variances and covariances of these errors are specified by a 7×7 space-time covariance matrix.

## Mathematical Analyses

### Statistical Performance Dispersion Model

Typical booster performance dispersions are shown in Table 1. Several six-degree-of-freedom (DOF) trajectory simulations are needed to determine the effect of these dispersions on position, velocity, and time errors at park orbit inject (end of velocity trim burn) and nominal payload deployment. In general, there are three different approaches available: 1) Monte Carlo simulation, 2) Taguchi's approach,[2] and 3) one-factor-at-a-time simulation.

The Monte Carlo simulation randomly selects the dispersion values within their 3-sigma limits and includes all dispersions in each simulation. Although this approach simulates a realistic flight situation, it requires a large number of expensive simulations.

In Taguchi's approach, one or more booster dispersions are simulated in a systematic order to account for possible interactions between various dispersions. This approach provides additional information for robust system design by controlling parameter values.

In one-factor-at-a-time simulations, an individual dispersion is set at ± 1-sigma value, while the other dispersions are

**Table 1 Commercial Titan class performance error sources**

| Error source | Description | 3-sigma value |
|---|---|---|
| $P_1$ | Stage 0 SRM[a] grain temperature | − 21.0°F |
| $P_2$ | Stage 0 Propellant weight | − 0.35% |
| $P_3$ | Stage 0 SRM web action time (WAT) | + 2.95% |
| $P_4$ | Stage 0 Thrust differential | + 3.0% |
| $P_5$ | Stage 0 SRM specific impulse | − 0.70% |
| $P_6$ | Stage 0 Dry weight | + 1.12% |
| $P_7$ | Stage 0 SRM ablatives | − 10.24% |
| $P_8$ | Stage 0 TVC[b] sideforce | − 10.0% during WAT<br>− 15.0% after WAT |
| $P_9$ | Stage 0 TVC dump | − 637.0 lb |
| $P_{10}$ | Stage 0 Thrust vector misalignment | 0.25 deg |
| $P_{11}$ | Stage I Thrust | + 4.08% |
| $P_{12}$ | Stage I Specific impulse | − 1.03% |
| $P_{13}$ | Stage I Thrust differential | + 2.85% |
| $P_{14}$ | Stage I Propellant weight | − 5330.0 lb |
| $P_{15}$ | Stage I Dry weight | + 297.0 lb |
| $P_{16}$ | Stage I Outage | + 1593.0 lb |
| $P_{17}$ | Stage II Thrust | + 4.54% |
| $P_{18}$ | Stage II Specific impulse | − 1.52% |
| $P_{19}$ | Stage II Tailoff impulse | + 5230.0 lb-s |
| $P_{20}$ | Stage II Propellant weight | − 1089.0 lb |
| $P_{21}$ | Stage II Dry weight | − 187.0 lb |
| $P_{22}$ | Axial force coefficient | + 10.0% |
| $P_{23}$ | Normal force coefficient | + 15.0% |
| $P_{24}$ | Payload fairing weight | + 100.0 lb |
| $P_{25}$ | IMU[c] misalignment in pitch | + 0.59 deg |
| $P_{26}$ | IMU misalignment in yaw | + 0.59 deg |
| $P_{27}$ | CG[d] offset in pitch axis | + 2.0 in. |
| $P_{28}$ | CG offset in yaw axis | + 2.0 in. |
| $P_{29}$ | Headwind profile | + 99.0% |
| $P_{30}$ | Sidewind profile | + 99.0% |
| $P_{31}$ | Autopilot gains | + 15.0% |
| $P_{32}$ | Atmospheric density | + 99.0% |

[a]Solid rocket motor
[b]Thrust vector control
[c]Inertial measurement unit
[d]Center of gravity

set at 0-sigma values. A total of 32 boost trajectory simulations are required for this approach for the booster dispersions shown in Table 1. To account for the nonsymmetric effect of the dispersion, either the plus or minus dispersion is simulated if it produces the larger dispersion. This classic conservative method requires a minimum number of simulations to determine the sensitivity of the booster dispersions.

### Unconditional Covariance Matrix Theory

The booster performance dispersions shown in Table 1 result in position, velocity, and time errors at the nominal deployment event. Therefore, the deployment time is stochastic in nature. An unconditional (variable time) 7 × 7 space-time covariance matrix represents the variances and covariances of these errors.

Let the space-time vector of the vehicle at payload deployment be represented by

$$\boldsymbol{x} = (x, y, z, v_x, v_y, v_z, t)' \tag{1}$$

and their mean (nominal) represented by

$$\boldsymbol{\mu_x} = (\mu_x, \mu_y, \mu_z, \mu_{v_x}, \mu_{v_y}, \mu_{v_z}, \mu_t)' \tag{2}$$

The deviations $(\boldsymbol{x} - \boldsymbol{\mu_x})$ are functions of the statistical performance parameters of the launch vehicle. If there are $n$ such statistical performance parameters, then

$$\boldsymbol{P} = (P_1, P_2, \dots, P_n)' \tag{3}$$

represents these parameters with their mean (nominal) values given by

$$\boldsymbol{\mu_P} = (\mu_{P_1}, \mu_{P_2}, \dots, \mu_{P_n}) \tag{4}$$

$(\boldsymbol{P} - \boldsymbol{\mu_P})$ represents the deviations of these parameters.

We can postulate a mathematical relationship between the 7 × 1 space-time deviations $(\boldsymbol{x} - \boldsymbol{\mu_x})$ and the $n \times 1$ parameter deviations $(\boldsymbol{P} - \boldsymbol{\mu_P})$ as

$$(\boldsymbol{x} - \boldsymbol{\mu_x}) = S(\boldsymbol{P} - \boldsymbol{\mu_P}) \tag{5}$$

where $S$ is a $7 \times n$ transformation matrix relating off-nominal space-time deviations to the off-nominal parameter deviations. Although the transformation from parameter deviation to space-time deviation is nonlinear, it can be linearized using Taylor's series, whereby

$$S = \begin{bmatrix} \dfrac{\partial x}{\partial P_1} & \dfrac{\partial x}{\partial P_2} & \cdots & \dfrac{\partial x}{\partial P_n} \\ \dfrac{\partial y}{\partial P_1} & \dfrac{\partial y}{\partial P_2} & \cdots & \dfrac{\partial y}{\partial P_n} \\ \cdot & \cdot & \cdots & \cdot \\ \dfrac{\partial t}{\partial P_1} & \dfrac{\partial t}{\partial P_2} & \cdots & \dfrac{\partial t}{\partial P_n} \end{bmatrix} \tag{6}$$

The matrix $S$ is known as the sensitivity matrix.

The first equation of the system of equations given by Eq. (5) is

$$(x - \mu_x) = \frac{\partial x}{\partial P_1}(P_1 - \mu_{P_1}) + \frac{\partial x}{\partial P_2}(P_2 - \mu_{P_2}) + \cdots + \frac{\partial x}{\partial P_n}(P_n - \mu_{P_n}) \tag{7}$$

which is a linear combination of a set of first-order truncated Taylor's series of off-nominal parameter deviations. Similar equations exist for $y$, $z$, $v_x$, $v_y$, $v_z$, and the time $t$.

Now consider that the performance dispersions are multivariate normally distributed with a mean (nominal) vector $\boldsymbol{\mu_P}$ and covariance matrix $\Sigma_P$. The performance deviations are $(\boldsymbol{P} - \boldsymbol{\mu_P})$, and the performance parameter covariance matrix $\Sigma_P$ is given by the expected value of the product of the deviation vector $(\boldsymbol{P} - \boldsymbol{\mu_P})$ with its transpose, as

$$\Sigma_P = \mathcal{E}[(\boldsymbol{P} - \boldsymbol{\mu_P})(\boldsymbol{P} - \boldsymbol{\mu_P})'] \tag{8}$$

Similarly, the space-time covariance matrix $\Sigma_x$ is given by the expected value of product of the space-time deviation vector $(\boldsymbol{x} - \boldsymbol{\mu_x})$ with its transpose, as

$$\Sigma_x = \mathcal{E}[(\boldsymbol{x} - \boldsymbol{\mu_x})(\boldsymbol{x} - \boldsymbol{\mu_x})'] \tag{9}$$

Using the linear transformation Eq. (5), it can be shown[3] that the space-time covariance matrix $\Sigma_x$ is related to the parameter covariance matrix $\Sigma_P$ by

$$\Sigma_x = S \Sigma_P S' \tag{10}$$

The matrix $\Sigma_x$ is an unconditional (variable time) 7 × 7 space-time covariance matrix of position, velocity, and time errors, whereas $\Sigma_P$ is a $n \times n$ performance parameter covariance matrix of parameters $P_1, P_2, \dots, P_n$. If the performance parameters are considered to be stochastically independent, then $\Sigma_P$

can be written as a $n \times n$ diagonal matrix with variances $\sigma^2_{P_1}$, $\sigma^2_{P_2}$, ..., $\sigma^2_{P_n}$ in the form

$$\Sigma_P = \begin{bmatrix} \sigma^2_{P_1} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \sigma^2_{P_2} & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \sigma^2_{P_n} \end{bmatrix} \tag{11}$$

The unconditional $7 \times 7$ space-time covariance matrix $\Sigma_x$ takes the form

$$\Sigma_x = \begin{bmatrix} \sigma^2_x & \sigma_{x,y} & \cdot & \cdot & \cdot & \sigma_{x,t} \\ \sigma_{y,x} & \sigma^2_y & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_{t,x} & \cdot & \cdot & \cdot & \cdot & \sigma^2_t \end{bmatrix} \tag{12}$$

Since the covariance matrix $\Sigma_x$ is symmetric, $\sigma_{x,y} = \sigma_{y,x}$, and so on.

For numerical computation purposes, Eq. (10) can be written as

$$\Sigma_x = S_* S_*' \tag{13}$$

with the $7 \times n$ pseudosensitivity matrix $S_*$ given as $S\Sigma_P^{1/2}$. The diagonal matrix $\Sigma_P^{1/2}$, with the performance parameter standard deviations along its diagonal, is the square root of $\Sigma_P$. The pseudosensitivity matrix $S_*$ can be computed by performing six-DOF simulations to determine the deviations in $x$, $y$, $z$, $v_x$, $v_y$, $v_z$, and $t$ at the payload deployment event for each of the 1-sigma performance parameter dispersions.

#### Conditional Covariance Matrix Theory

The unconditional (variable time) covariance matrix, which includes the space-time variates ($x$, $y$, $z$, $v_x$, $v_y$, $v_z$, and $t$), is presented in Eq. (12). To determine the orbital elements dispersions, it is necessary to compute a conditional (fixed time) covariance matrix, which includes the space variables ($x$, $y$, $z$, $v_x$, $v_y$, $v_z$), about a nominal event time. Therefore, a method is required to compute a conditional covariance matrix at a specified time from a given unconditional covariance matrix. This is achieved by using the conditional distribution theory of statistics.[3]

Consider a $p \times 1$ vector random variable denoted by $\boldsymbol{x}$. We can partition this vector into two subvectors: 1) a vector $\boldsymbol{x}_{(1)}$ of dimension $(p-q) \times 1$, and 2) a vector $\boldsymbol{x}_{(2)}$ of dimension $q \times 1$. This is represented by

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_{(1)} \\ \hline \boldsymbol{x}_{(2)} \end{bmatrix} \tag{14}$$

The mean (nominal) values of $\boldsymbol{x}_{(1)}$ and $\boldsymbol{x}_{(2)}$ are denoted by $\mu_{x_{(1)}}$ and $\mu_{x_{(2)}}$, which are $(p-q) \times 1$ and $q \times 1$ subvectors of the $p \times 1$ mean vector $\mu_x$ of the vector random variable $\boldsymbol{x}$. In other words, it is represented by

$$\mu_x = \begin{bmatrix} \mu_{x_{(1)}} \\ \hline \mu_{x_{(2)}} \end{bmatrix} \tag{15}$$

The unconditional covariance matrix $\Sigma_x$ is also partitioned as follows:

$$\Sigma_x = \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right] \tag{16}$$

where

$\Sigma_{11}$ = a covariance matrix of a size $(p-q) \times (p-q)$
$\Sigma_{12}$ = a covariance matrix of a size $(p-q) \times q$
$\Sigma_{21}$ = a covariance matrix of a size $q \times (p-q) = \Sigma_{12}'$ by symmetry
$\Sigma_{22}$ = a covariance matrix of a size $q \times q$

If $\boldsymbol{x}$ has a multivariate normal distribution according to $N[\mu_x, \Sigma_x]$, then based on the conditional theory of statistical distributions, the $p-q$ dimensional conditional probability density function of $\boldsymbol{x}_{(1)}$, given that $\boldsymbol{x}_{(2)}$ has some particular value (denoted by $\alpha_{(2)}$), is

$$N[\mu_{x_{(1)}} + \Sigma_{12}\Sigma_{22}^{-1}(\alpha_{(2)} - \mu_{x_{(2)}}),\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}] \tag{17}$$

The distribution given by Eq. (17) shows that 1) the mean of the $p-q$ conditional probability density function depends only linearly on the variates held fixed, and 2) the variances and covariances do not depend at all on the values of the fixed variates in $\alpha_{(2)}$.

If the fixed variates of $\alpha_{(2)}$ are set at their mean values, then the conditional probability function reduces to

$$N[\mu_{x_{(1)}},\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}] \tag{18}$$

which shows that the conditional mean vector of the $(p-q) \times 1$ vector random variable $\boldsymbol{x}_{(1)}$ is the same as the unconditional mean vector, and the conditional covariance matrix $\Sigma_{11\cdot 2}$ is given by

$$\Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\,\Sigma_{22}^{-1}\,\Sigma_{21} \tag{19}$$

which has a size of $(p-q) \times (p-q)$.

Referring to the space-time variables, the unconditional $7 \times 7$ space-time covariance matrix is $\Sigma_x$, which consists of four partioned matrices: 1) an unconditional $6 \times 6$ space covariance matrix $\Sigma_{11}$, 2) a $6 \times 1$ covariance matrix $\Sigma_{12}$, 3) a $1 \times 6$ covariance matrix $\Sigma_{21}$, which is equal to $\Sigma_{12}'$ by the symmetry of statistical covariance matrices, and 4) a $1 \times 1$ scalar variance ($= \Sigma_{22}$) of the time deviations. The conditional $6 \times 6$ space covariance matrix is given by Eq. (19) where

$$\Sigma_{11} = \begin{bmatrix} \sigma^2_x & \sigma_{x,y} & \cdot & \cdot & \cdot & \sigma_{x,v_z} \\ \sigma_{y,x} & \sigma^2_y & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_{v_z,x} & \cdot & \cdot & \cdot & \cdot & \sigma^2_{v_z} \end{bmatrix} \tag{20}$$

$$\Sigma_{12} = \begin{bmatrix} \sigma_{x,t} \\ \sigma_{y,t} \\ \sigma_{z,t} \\ \sigma_{v_x,t} \\ \sigma_{v_y,t} \\ \sigma_{v_z,t} \end{bmatrix} \tag{21}$$

$$\Sigma_{21} = \Sigma_{12}' \tag{22}$$

$$\Sigma_{22} = \sigma_t^2 \tag{23}$$

The conditional probability density function of $x_{(1)}$ [ = $(x, y, z, v_x, v_y, v_z)'$], given that $x_{(2)}$ has a particular value $\alpha_{(2)} = t$, is

$$N[\mu_{x_{(1)}} + \Sigma_{12}\, \Sigma_{22}^{-1}(t - \mu_t),\ \Sigma_{11} - \Sigma_{12}\, \Sigma_{22}^{-1}\Sigma_{21}] \tag{24}$$

**Higher-Order Covariance Matrix Theory**

By including the time deviations in the unconditional $7 \times 7$ space-time covariance matrix theory, we establish a first-order linear relationship between the space variable $x_{(1)}$ [ = $(x, y, z, v_x, v_y, v_z)'$] and the time variable $x_{(2)}$ ( = $t$). The form of this relationship is given by

$$x_{(1)} - \mu_{x_{(1)}} = \beta(t - \mu_t) \tag{25}$$

or

$$\Delta x_{(1)} = \beta(\Delta t)$$

where

$$\Delta x_{(1)} = x_{(1)} - \mu_{x_{(1)}}$$

$$\Delta t = (t - \mu_t)$$

The matrix $\beta$ is the linear regression coefficient matrix, which is of a size $6 \times 1$ for the first-order time model. The matrix $\beta$ is given as a function of the partitions of $\Sigma_x$ as

$$\beta = \Sigma_{12}\Sigma_{22}^{-1} \tag{26}$$

Now, consider the motion of a spacecraft in a Keplerian orbit subjected to a spherically symmetric gravitational field. If we know the position $r_0$ and the velocity $\mathrm{d}r_0/\mathrm{d}t$ at a given time $t_0$, then the position and velocity vectors at anytime $t( = t_0 + \Delta t)$ can be expressed as two Taylor's series,[4] given by

$$r_t = r_0 + \frac{\mathrm{d}r_0}{\mathrm{d}t}(\Delta t) + \frac{1}{2!}\frac{\mathrm{d}^2 r_0}{\mathrm{d}t^2}(\Delta t)^2 + \cdots + o(r_0, \frac{\mathrm{d}r_0}{\mathrm{d}t}, \Delta t) \tag{27}$$

and

$$\frac{\mathrm{d}r_t}{\mathrm{d}t} = \frac{\mathrm{d}r_0}{\mathrm{d}t} + \frac{\mathrm{d}^2 r_0}{\mathrm{d}t^2}(\Delta t) + \frac{1}{2!}\frac{\mathrm{d}^3 r_0}{\mathrm{d}t^3}(\Delta t)^2 + \cdots + o(r_0, \frac{\mathrm{d}r_0}{\mathrm{d}t}, \Delta t) \tag{28}$$

The second-order derivative of $r_0$ ( = $\mathrm{d}^2 r_0/\mathrm{d}t^2$) is given, for the spherically symmetric Newtonian gravitational field, as

$$\frac{\mathrm{d}^2 r_0}{\mathrm{d}t^2} = -\frac{G}{\|r_0\|^3} r_0 \tag{29}$$

where $G$ is Earth's gravitational constant. By differentiating Eq. (29), we compute the higher-ordered ( $\geqq 3$) derivatives of Eqs. (27) and (28), and they are functions of only $r_0$ and $\mathrm{d}r_0/\mathrm{d}t$.

Equations (27) and (28) for the position and velocity of the spacecraft at any time $t$ can be written in the polynomial regression form using the space variable $x_{(1)}$[ = $(x, y, z, v_x, v_y, v_z)'$], as

$$x_{(1)} = \beta_0 + \beta_1(\Delta t) + \beta_2(\Delta t)^2 + \cdots + o(\beta_0, \Delta t) \tag{30}$$

where $\beta_0, \beta_1, \cdots$ are polynomial regression coefficient vectors.

Neter and Wasserman[5] have shown that polynomial regression theory is a special case of multiple regression theory. We can define new variables $t_1$, $t_2$, $t_3$, etc., such that $t_1 = \Delta t$, $t_2 = (\Delta t)^2$, $t_3 = (\Delta t)^3$, etc. The regression model given by Eq. (30) can then be expressed as

$$x_{(1)} = \beta_0 + \beta_1 t_1 + \beta_2 t_2 + \cdots + o(\beta_0, t_1) \tag{31}$$

which is of the classic form of multiple regression models. Models defined in Eq. (30) are known[5] as "intrinscally linear." Therefore, we can easily include the higher-order time effects in the unconditional covariance matrix theory by considering $\Delta t$, $(\Delta t)^2$, etc., as "independent" variables.

To develop higher-ordered unconditional covariance matrices, the space-time vector random variable $x$ is defined to include higher-ordered time effects. For example, considering third-order time effects, the stochastic space-time vector random variable is defined to be the $9 \times 1$ vector $x = (x, y, z, v_x, v_y, v_z, t_1, t_2, t_3)'$. The $9 \times 1$ mean (nominal) vector of $x$ is defined as

$$\mu_x = (\mu_x, \mu_y, \mu_z, \mu_{v_x}, \mu_{v_y}, \mu_{v_z}, \mu_{t_1}, \mu_{t_2}, \mu_{t_3})' \tag{32}$$

The $9 \times 9$ unconditional covariance matrix is obtained from

$$\Sigma_x = S\Sigma_P S' \tag{33}$$

where the $9 \times n$ sensitivity matrix $S$ is defined as

$$S = \begin{bmatrix} \frac{\partial x}{\partial P_1} & \frac{\partial x}{\partial P_2} & \cdots & \frac{\partial x}{\partial P_n} \\ \frac{\partial y}{\partial P_1} & \frac{\partial y}{\partial P_2} & \cdots & \frac{\partial y}{\partial P_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial t_1}{\partial P_1} & \frac{\partial t_1}{\partial P_2} & \cdots & \frac{\partial t_1}{\partial P_n} \\ \frac{\partial t_2}{\partial P_1} & \frac{\partial t_2}{\partial P_2} & \cdots & \frac{\partial t_2}{\partial P_n} \\ \frac{\partial t_3}{\partial P_1} & \frac{\partial t_3}{\partial P_2} & \cdots & \frac{\partial t_3}{\partial P_n} \end{bmatrix} \tag{34}$$

The matrix $S$ is a matrix of first-order derivatives of $x$, $y$, $z$, $v_x$, $v_y$, $v_z$, $t_1$, $t_2$, and $t_3$ with respect to the various performance parameters. The $9 \times 9$ unconditional covariance matrix defined by $\Sigma_x$ includes the polynomial regressional relationships between the space variables $x$, $y$, $z$, $v_x$, $v_y$, $v_z$ and the time variables $t_1$, $t_2$, and $t_3$. The conditional $6 \times 6$ space covariance matrix can be obtained using Eq. (19), which requires the computation of the inverse of the $3 \times 3$ matrix $\Sigma_{22}$. With the addition of second- or third-order time effects, the conditional $6 \times 6$ space covariance matrix converges rapidly and provides a better comparison to the numerically computed conditional space covariance matrix.

**Variance Equality Hypothesis Test**

To compare various conditional space covariance matrices for convergence and equality, hypothesis tests are performed. Since the variances define the spacecraft position and velocity error ranges, variance equality tests are performed.

For our application, we will pairwise test for the variance equality between the numerically and theoretically computed $6 \times 6$ conditional covariance matrices. We will also pairwise test for the variance equality between two theoretically extracted $6 \times 6$ conditional matrices, with each being extracted from a different sized unconditional (variable time) covariance matrix. For each pairwise test, the hypothesis to test is

$$H_o: \sigma_1^2 = \sigma_2^2 \tag{35}$$

where $\sigma_1^2$ is the theoretically computed variance and $\sigma_2^2$ the numerically (or another theoretically) computed variance. For each matrix to matrix comparison, we will perform six pairwise hypothesis tests.

Several variance equality hypothesis tests such as Hartley's test,[6] Cochran's test,[7] the generalized likelihood-ratio test,[6] and Bartlett's test[5] are available. The Hartley test, which is applicable to equal sample sizes, is based on the ratio between the largest and the smallest variances. It is the simplest test for testing the equality of variances. Cochran's test is similar to Hartley's test and is based on the ratio of the largest variance and of the sum of all variances. The generalized likelihood-ratio test computes the likelihood function to determine the equality of variances.

The Bartlett test is an all-purpose test and can be used for different sample sizes. It is based on the comparison between the weighted arithmetic and geometric mean square errors. Bartlett has shown that the difference between the logarithmic value of arithmetic and geometric mean square errors follows approximately the chi-square ($\chi^2$) distribution when the population variances are equal. The test can be summarized as follows. Compute

$$B = \frac{2.302585}{C}\left[(N-k)\log s_p^2 - \sum_{i=1}^{k}(n_i - 1)\log s_i^2\right] \tag{36}$$

where $k$ is the number of variances to test, $N$ is the total sample size, and $n_i$ is the sample size used for the $i$th variance estimate $s_i^2$,

$$C = 1 + \frac{1}{3(k-1)}\left[\sum_{i=1}^{k}\frac{1}{n_i - 1} - \frac{1}{N-k}\right]$$

$$s_p^2 = \frac{1}{N-k}\sum_{i=1}^{k}(n_i - 1)s_i^2 = \text{the arithmetic mean square error}$$

For each pairwise variance comparison, $k = 2$, $n_i = 32$, and $N = 64$.

**Table 2 Transpose of the unconditional pseudosensitivity martix ($S'_*$)**

| Error source | $\Delta x$ | $\Delta y$ | $\Delta z$ | $\Delta v_x$ | $\Delta v_y$ | $\Delta v_z$ | $\Delta t$ |
|---|---|---|---|---|---|---|---|
| $P_1$ | 2.31371E + 03 | − 2.74604E + 04 | 1.50231E + 04 | 3.46184E + 01 | 1.88497E + 00 | 2.53733E − 02 | 2.273E + 00 |
| $P_2$ | 2.39831E + 03 | − 2.70613E + 04 | 1.48117E + 04 | 3.41917E + 01 | 1.74838E + 00 | 9.11133E − 02 | 1.293E + 00 |
| $P_3$ | − 1.03043E + 02 | − 3.20010E + 02 | 1.40343E + 02 | 2.80943E − 01 | 1.42230E − 01 | − 7.73000E − 02 | 1.173E + 00 |
| $P_4$ | 3.83750E + 03 | − 4.27950E + 04 | 2.34584E + 04 | 5.40796E + 01 | 2.84260E + 00 | 1.13720E − 01 | 2.020E + 00 |
| $P_5$ | 2.28449E + 03 | − 2.56204E + 04 | 1.40532E + 04 | 3.23991E + 01 | 1.64241E + 00 | 1.08320E − 01 | 1.347E + 00 |
| $P_6$ | 3.36538E + 03 | − 3.76652E + 04 | 2.06011E + 04 | 4.75783E + 01 | 2.47121E + 00 | 9.91633E − 02 | 1.767E + 00 |
| $P_7$ | 3.64084E + 03 | − 4.07374E + 04 | 2.22922E + 04 | 5.14649E + 01 | 2.69460E + 00 | 1.01477E − 01 | 1.893E + 00 |
| $P_8$ | − 2.24830E + 02 | 2.64645E + 03 | − 1.43057E + 03 | − 3.34378E + 00 | − 1.63743E − 01 | − 1.25867E − 02 | − 1.400E − 01 |
| $P_9$ | 3.47179E + 03 | − 3.89785E + 04 | 2.13196E + 04 | 4.92323E + 01 | 2.57694E + 00 | 9.54367E − 02 | 1.753E + 00 |
| $P_{10}$ | 4.53261E + 03 | − 5.03004E + 04 | 2.75249E + 04 | 6.35528E + 01 | 3.36866E + 00 | 9.74467E − 02 | 2.273E + 00 |
| $P_{11}$ | 2.95514E + 03 | − 4.21198E + 04 | 2.30073E + 04 | 5.27273E + 01 | 3.53671E + 00 | − 3.34787E − 01 | 3.373E + 00 |
| $P_{12}$ | − 3.26308E + 03 | 4.05280E + 04 | − 2.21541E + 04 | − 5.11268E + 01 | − 2.46395E + 00 | − 2.09627E − 01 | − 1.787E + 00 |
| $P_{13}$ | 8.08553E + 02 | − 9.35589E + 03 | 5.14495E + 03 | 1.18224E + 01 | 6.02483E − 01 | 4.31100E − 02 | 4.267E − 01 |
| $P_{14}$ | 5.25407E + 02 | − 3.65037E + 03 | 2.01492E + 03 | 4.74233E + 00 | 3.58800E − 02 | 1.30350E − 01 | − 2.867E − 01 |
| $P_{15}$ | 1.61725E + 03 | − 1.84560E + 04 | 1.00974E + 04 | 2.33133E + 01 | 1.18212E + 00 | 6.57300E − 02 | 9.067E − 01 |
| $P_{16}$ | 1.63797E + 02 | − 2.11499E + 03 | 1.12644E + 03 | 2.65158E + 00 | 1.43870E − 01 | − 5.94667E − 03 | 2.533E − 01 |
| $P_{17}$ | 4.08270E + 03 | − 6.82882E + 04 | 3.72598E + 04 | 8.49701E + 01 | 6.67348E + 00 | − 1.07359E + 00 | 3.700E + 00 |
| $P_{18}$ | 2.49233E + 02 | 1.04876E + 03 | − 5.81454E + 02 | − 1.14106E + 00 | − 3.82000E − 01 | 1.67157E − 01 | − 8.667E − 02 |
| $P_{19}$ | − 1.21803E + 04 | 1.70865E + 05 | − 9.34542E + 04 | − 2.15665E + 02 | − 8.17806E + 00 | − 2.12748E + 00 | − 7.827E + 00 |
| $P_{20}$ | 1.32671E + 03 | − 9.76099E + 03 | 5.35434E + 03 | 1.25971E + 01 | 1.77907E − 01 | 2.90920E − 01 | 9.333E − 02 |
| $P_{21}$ | 1.51752E + 03 | − 1.82137E + 04 | 9.97037E + 03 | 2.29608E + 01 | 1.24129E + 00 | 2.65467E − 02 | 8.933E − 01 |
| $P_{22}$ | 2.35836E + 03 | − 2.64611E + 04 | 1.44776E + 04 | 3.34402E + 01 | 1.69324E + 00 | 9.70833E − 02 | 1.373E + 00 |
| $P_{23}$ | 8.05437E + 02 | − 9.40857E + 03 | 5.12990E + 03 | 1.18620E + 01 | 6.04170E − 01 | 2.56267E − 02 | 4.267E − 01 |
| $P_{24}$ | 8.35600E + 02 | − 9.64212E + 03 | 5.27471E + 03 | 1.21696E + 01 | 6.13783E − 01 | 3.58667E − 02 | 4.600E − 01 |
| $P_{25}$ | − 3.57767E + 02 | 9.41571E + 03 | − 5.12779E + 03 | − 1.16805E + 01 | − 9.91860E − 01 | 1.89777E − 01 | − 3.400E − 01 |
| $P_{26}$ | 1.26542E + 03 | − 1.45819E + 04 | 7.94360E + 03 | 1.83901E + 01 | 9.31497E − 01 | 4.09667E − 02 | 6.667E − 01 |
| $P_{27}$ | 1.01116E + 03 | − 1.13517E + 04 | 6.25048E + 03 | 1.43569E + 01 | 7.10707E − 01 | 6.75200E − 02 | 5.200E − 01 |
| $P_{28}$ | 4.86960E + 03 | − 5.37872E + 04 | 2.94158E + 04 | 6.79409E + 01 | 3.61224E + 00 | 9.35367E − 02 | 2.400E + 00 |
| $P_{29}$ | 5.18582E + 03 | − 5.64434E + 04 | 3.09137E + 04 | 7.13640E + 01 | 3.75276E + 00 | 1.36677E − 01 | 2.767E + 00 |
| $P_{30}$ | 1.94056E + 03 | − 2.18903E + 04 | 1.19720E + 04 | 2.76537E + 01 | 1.39391E + 00 | 8.11667E − 02 | 1.027E + 00 |
| $P_{31}$ | − 2.34797E + 02 | 2.80214E + 03 | − 1.54736E + 03 | − 3.56325E + 00 | − 1.77773E − 01 | − 1.59733E − 02 | − 1.333E − 01 |
| $P_{32}$ | 3.84720E + 02 | − 4.55956E + 03 | 2.47700E + 03 | 5.74369E + 00 | 2.92217E − 01 | 9.05333E − 03 | 2.067E − 01 |

**Table 3 7×7 unconditional covariance matrix ($S_* S'_*$)**

| | $x$ | $y$ | $z$ | $v_x$ | $v_y$ | $v_z$ | $t$ |
|---|---|---|---|---|---|---|---|
| $x$ | 3.446319514E+08 | −4.397717398E+09 | 2.405500483E+09 | 5.546920320E+06 | 2.625494404E+05 | 2.601032708E+04 | 2.113219046E+05 |
| $y$ | −4.397717398E+09 | 5.715647311E+10 | −3.126089447E+10 | −7.206372321E+07 | −3.416400053E+06 | −3.344658255E+0 | −2.760092044E+06 |
| $z$ | 2.405500483E+09 | −3.126089447E+10 | 1.709771125E+10 | 3.941434888E+07 | 1.868309445E+06 | 1.830740327E+05 | 1.509466315E+06 |
| $v_x$ | 5.546920320E+06 | −7.206372321E+07 | 3.941434888E+07 | 9.086080903E+04 | 4.303675468E+03 | 4.238548559E+02 | 3.478539661E+03 |
| $v_y$ | 2.625494404E+05 | −3.416400053E+06 | 1.868309445E+06 | 4.303675468E+03 | 2.174439599E+02 | 1.255615857E+01 | 1.686395724E+02 |
| $v_z$ | 2.601032708E+04 | −3.344658255E+05 | 1.830740327E+05 | 4.238548559E+02 | 1.255615857E+02 | 6.137732991E+00 | 1.406644184E+01 |
| $t$ | 2.113219046E+05 | −2.760092044E+06 | 1.509466315E+06 | 3.478539661E+03 | 1.686395724E+02 | 1.406644184E+01 | 1.382011111E+02 |

Table 4 Transpose of the conditional pseudosensitivity martix ($S_*'$)

| Error source | $\Delta x$ | $\Delta y$ | $\Delta z$ | $\Delta v_x$ | $\Delta v_y$ | $\Delta v_z$ |
|---|---|---|---|---|---|---|
| $P_1$ | −1.95294E+03 | 2.17692E+04 | −1.19246E+04 | −2.76501E+01 | −1.19330E+00 | −1.96860E−01 |
| $P_2$ | −7.94633E+01 | 9.44078E+02 | −5.19664E+02 | −1.23646E+00 | −5.11800E−02 | −9.10333E−03 |
| $P_3$ | −2.23735E+03 | 2.50921E+04 | −1.37679E+04 | −3.18555E+01 | −1.38132E+00 | −2.27667E−01 |
| $P_4$ | −9.26200E+01 | 9.42212E+02 | −4.87225E+02 | −1.25554E+00 | −2.60900E−02 | −1.07867E−02 |
| $P_5$ | −2.85263E+02 | 3.54048E+03 | −1.91042E+03 | −4.48984E+00 | −2.21447E−01 | −1.19667E−03 |
| $P_6$ | −5.57933E+01 | 5.87781E+02 | −3.41323E+02 | −8.16754E−01 | −2.23100E−02 | −1.84400E−02 |
| $P_7$ | −3.78433E+01 | 2.57476E+02 | −1.51793E+02 | −4.00222E−01 | 1.05500E−02 | −1.80833E−02 |
| $P_8$ | 3.58633E+01 | −3.85422E+02 | 2.28982E+02 | 4.90987E−01 | 2.38067E−02 | 2.19333E−03 |
| $P_9$ | 6.67767E+01 | −1.01486E+03 | 5.35228E+02 | 1.20247E+00 | 9.29233E−02 | −1.61933E−02 |
| $P_{10}$ | 6.86567E+01 | −1.08029E+03 | 5.76328E+02 | 1.27727E+00 | 1.00640E−01 | −2.14467E−02 |
| $P_{11}$ | −3.40689E+03 | 3.09217E+04 | −1.69756E+04 | −3.96543E+01 | −1.06378E+00 | −6.46420E−01 |
| $P_{12}$ | −7.13800E+01 | 1.82868E+03 | −9.75459E+02 | −2.19123E+00 | −2.01047E−01 | 5.01567E−02 |
| $P_{13}$ | 4.58000E+00 | −1.16343E+02 | 8.72203E+01 | 1.35214E−01 | 2.17733E−02 | 3.05333E−03 |
| $P_{14}$ | 1.06789E+03 | −9.85823E+03 | 5.41317E+03 | 1.25951E+01 | 4.28193E−01 | 1.56117E−01 |
| $P_{15}$ | −1.04560E+02 | 1.17740E+03 | −6.50356E+02 | −1.52245E+00 | −6.47967E−02 | −1.24467E−02 |
| $P_{16}$ | −3.08397E+02 | 3.37124E+03 | −1.87655E+03 | −4.28751E+00 | −1.95963E−01 | −3.24833E−02 |
| $P_{17}$ | −3.20382E+03 | 1.17960E+04 | −6.58757E+03 | −1.63292E+01 | 1.32448E+00 | −1.04956E+00 |
| $P_{18}$ | 4.11000E+02 | −8.28179E+02 | 4.45933E+02 | 1.23306E+00 | −2.65563E−01 | 1.76140E−01 |
| $P_{19}$ | −1.21337E+02 | 1.26374E+03 | −6.96800E+02 | −1.38318E+00 | −1.14693E−01 | 1.86900E−02 |
| $P_{20}$ | 1.14934E+03 | −7.73979E+03 | 4.24789E+03 | 1.00401E+01 | 4.94833E−02 | 2.82900E−01 |
| $P_{21}$ | −1.78513E+02 | 1.13080E+03 | −6.19184E+02 | −1.50915E+00 | 1.30633E−02 | −5.06333E−02 |
| $P_{22}$ | −2.65070E+02 | 3.27711E+03 | −1.80216E+03 | −4.17929E+00 | −2.10290E−01 | −1.32733E−02 |
| $P_{23}$ | 1.41333E+00 | −1.69030E+02 | 7.21917E+01 | 1.74823E−01 | 2.33800E−02 | −1.44533E−02 |
| $P_{24}$ | −3.10333E+01 | 3.19279E+02 | −1.78141E+02 | −4.30661E−01 | −1.21667E−02 | −7.43333E−03 |
| $P_{25}$ | 2.69637E+02 | 2.05188E+03 | −1.09724E+03 | −2.36661E+00 | −5.42013E−01 | 2.28780E−01 |
| $P_{26}$ | 2.65667E+00 | −1.45395E+02 | 4.09128E+01 | 1.28713E−01 | 1.77500E−02 | −1.82867E−02 |
| $P_{27}$ | 2.93567E+01 | −9.10897E+01 | 8.63349E+01 | 1.12948E−01 | 1.09000E−03 | 1.97433E−02 |
| $P_{28}$ | 1.37823E+02 | −1.82571E+03 | 9.65725E+02 | 2.19494E+00 | 1.43557E−01 | −2.20167E−02 |
| $P_{29}$ | −2.55593E+02 | 3.45772E+03 | −1.88311E+03 | −4.42716E+00 | −2.32720E−01 | −3.31667E−03 |
| $P_{30}$ | −1.74500E+01 | 3.41273E+02 | −1.98207E+02 | −4.69480E−01 | −2.60533E−02 | −3.01667E−03 |
| $P_{31}$ | 1.33567E+01 | −8.53670E+01 | 3.31625E+01 | 8.89070E−02 | 7.33333E−04 | −1.82333E−03 |
| $P_{32}$ | −2.80333E+00 | −8.40633E+01 | 2.71794E+01 | 8.27740E−02 | 1.27567E−02 | −1.13700E−02 |

Table 5 Numerically computed 6×6 conditional covariance matrix ($S_* S_*'$)

| | $x$ | $y$ | $z$ | $v_x$ | $v_y$ | $v_z$ |
|---|---|---|---|---|---|---|
| $x$ | 3.381999715E+07 | −2.660468496E+08 | 1.463371321E+08 | 3.439285089E+05 | 5.374208175E+03 | 7.102606517E+03 |
| $y$ | −2.660468496E+08 | 2.423169328E+09 | −1.330548556E+09 | −3.106515512E+06 | −8.749742352E+04 | −4.585205331E+04 |
| $z$ | 1.463371321E+08 | −1.330548556E+09 | 7.306218980E+08 | 1.705937942E+06 | 4.781242016E+04 | 2.529355561E+04 |
| $v_x$ | 3.439285089E+05 | −3.106515512E+06 | 1.705937942E+06 | 3.984453771E+03 | 1.095253480E+02 | 6.007842117E+01 |
| $v_y$ | 5.374208175E+03 | −8.749742352E+04 | 4.781242016E+04 | 1.095253480E+02 | 7.057923694E+00 | −2.512583638E−01 |
| $v_z$ | 7.102606517E+03 | −4.585205331E+04 | 2.529355561E+04 | 6.007842117E+01 | −2.512583638E−01 | 1.807834828E+00 |

Table 6 Theoretical 6×6 conditional covariance matrix computed from the 7×7 unconditional covariance matrix

| | $x$ | $y$ | $z$ | $v_x$ | $v_y$ | $v_z$ |
|---|---|---|---|---|---|---|
| $x$ | 2.150178979E+07 | −1.772889010E+08 | 9.739098897E+07 | 2.279209246E+05 | 4.684396141E+03 | 4.501474822E+03 |
| $y$ | −1.772889010E+08 | 2.033124023E+09 | −1.114494535E+09 | −2.591853073E+06 | −4.840439522E+04 | −5.353701168E+04 |
| $z$ | 9.739098897E+07 | −1.114494535E+09 | 6.109512089E+08 | 1.420888467E+06 | 2.638681581E+04 | 2.943691676E+04 |
| $v_x$ | 2.279209246E+05 | −2.591853073E+06 | 1.420888467E+06 | 3.305520387E+03 | 5.899583888E+01 | 6.980071379E+01 |
| $v_y$ | 4.684396141E+03 | −4.840439522E+04 | 2.638681581E+04 | 5.899583888E+01 | 1.166192853E+01 | −4.608383149E+00 |
| $v_z$ | 4.501474822E+03 | −5.353701168E+04 | 2.943691676E+04 | 6.980071379E+01 | −4.608383149E+00 | 4.706016672E+00 |

The quantity $B$ is large when the two variances differ greatly, and $B$ is equal to zero when the two variances are equal. If $B \leqq \chi^2(1-\alpha;\ k-1)$, we conclude that there is no significant difference between the variances, and the hypothesis holds true. The quantity $\chi^2(1-\alpha;\ k-1)$ is the $(1-\alpha)$ 100% confidence level of the $\chi^2$ distribution with $k-1$ degrees of freedom. The values of $\chi^2$ for a given $\alpha$ and degrees of freedom can be found in statistical text books.[5,6]

## Numerical Results

To validate the analysis method, six-DOF trajectory simulation runs were made for the Commercial Titan/dual payload mission from liftoff through the payload one separation event. The 3-sigma booster performance dispersions used for the simulation runs are shown in Table 1. Since the velocity trim burn corrects the stage II shutdown impulse uncertainty by varying the duration of the trim burn, a variation of the payload one deployment time will be noticed. For the actual Commercial Titan flights this deployment time variation is significantly reduced by using an inflight nodal crossing prediction algorithm so that the payload is deployed at a specified time from an ascending or descending node.

A total of 32 trajectory simulations, each with one dispersion, are performed to generate the pseudosensitivity matrix at the nominal payload deployment event as shown in Table 2. (Please note that in all of the tables to follow, the units are seconds for time, feet for positions, and feet/second for velocities.) Table 3 presents an unconditional 7 × 7 covariance matrix of position and velocity errors in Earth-Centered Inertial

(ECI) frame and of time errors, which is obtained by multiplying the pseudosensitivity matrix by its transpose.

The performance dispersion simulation runs also provide a pseudosensitivity matrix at a nominal payload deployment time (Table 4), which produces no time errors. A conditional covariance matrix of position and velocity errors is numerically generated by multiplying this pseudosensitivity matrix by its transpose, the computation resulting in Table 5.

The multivariate conditional distribution theory provides a technique for converting an unconditional (variable-time) covariance matrix (Table 3) into a conditional (fixed-time) covariance matrix (Table 6). The Bartlett test for equality of the variances between the theoretically computed conditional covariance matrix (Table 6) and the numerically computed conditional covariance matrix (Table 5) shows that there is a significant difference between the $v_z$ velocity variances at a 95% confidence level. The results of this chi-square hypothesis test are presented in Table 7, column 1. The significant difference is due to the linear first-order regression relationship between the state vector and time inherent in the $7 \times 7$ unconditional covariance matrix. This difference demonstrates that a linear first-order regression model is inadequate for a spacecraft traveling in a Keplerian orbit.

A higher-order polynomial regressional relationship between the state vector and time is required to account for the gravitational acceleration. A higher-order unconditional covariance matrix of a size $8 \times 8$ or $9 \times 9$ is needed to compute the $6 \times 6$ conditional covariance matrix. Table 8 presents the unconditional $8 \times 8$ covariance matrix. It is derived from the pseudosensitivity matrix by adding a $(\Delta t)^2$ column to Table 2 to account for the delta changes in $t^2$ at the payload deployment event for each performance dispersion. A conditional covariance matrix, shown in Table 9, is computed using the partitioned matrices shown in Table 8. The hypothesis tests of variance equality (Table 7, column 2) between the conditional covariance matrix (Table 6) derived from the $7 \times 7$ unconditional covariance matrix, and the conditional covariance matrix (Table 9) derived from the $8 \times 8$ unconditional covariance matrix show that there is a significant difference between the variance in the $v_z$ variable at the 95% confidence level. We must therefore include at least the second-order time effects, and we must continue to test for a significant third-order time effect.

To investigate the effect of a third-order polynomial regressional relationship between the state vector elements and time, a $9 \times 9$ unconditional covariance matrix (Table 10) and its corresponding conditional $6 \times 6$ covariance matrix (Table 11) are computed. Bartlett's equality of variance tests show that there are no significant differences in the state variances between the conditional matrices (Tables 11 and 9) obtained from $9 \times 9$ and $8 \times 8$ unconditional matrices, respectively. The results of the hypothesis tests are given in Table 7, column 3.

A convergence has taken place by having included the $(\Delta t)^2$ and $(\Delta t)^3$ terms in the linear regression relationship. Since there are no significant differences between the second- and third-order models, we choose the second-order model as the correct polynomial.

The adequacy of including second-order time effects in the polynomial regressional relationship to account for the gravitational acceleration of a spacecraft in a Keplerian orbit is demonstrated by the hypothesis tests of variance equality (Table 7, column 4) between the theoretically computed conditional covariance matrix (Table 9) obtained from the $8 \times 8$ unconditional matrix and the numerically computed conditional covariance matrix (Table 5). The variances are equal at the 95% confidence level. We therefore conclude that the numerical computation is adequate.

**Table 7 Chi-square ($\chi^2$) values for Bartlett's tests[a] of variance equalities**

| | | Col. 1[b] | Col. 2[c] | Col. 3[d] | Col. 4[e] |
|---|---|---|---|---|---|
| $H_o: (\sigma_x^2)_1$ | $= (\sigma_x^2)_2$ | 1.551 | 0.002 | 0.483 | 1.676 |
| $H_o: (\sigma_y^2)_1$ | $= (\sigma_y^2)_2$ | 0.235 | 0.487 | 0.019 | 1.390 |
| $H_o: (\sigma_z^2)_1$ | $= (\sigma_z^2)_2$ | 0.244 | 0.486 | 0.020 | 1.410 |
| $H_o: (\sigma_{v_x}^2)_1$ | $= (\sigma_{v_x}^2)_2$ | 0.266 | 0.506 | 0.025 | 1.497 |
| $H_o: (\sigma_{v_y}^2)_1$ | $= (\sigma_{v_y}^2)_2$ | 1.904 | 1.086 | 0.005 | 0.117 |
| $H_o: (\sigma_{v_z}^2)_1$ | $= (\sigma_{v_z}^2)_2$ | 6.729[f] | 15.010[f] | 1.475 | 1.921 |

[a]Degree of freedom for each test equals 1. For every test, the distributional $\chi^2(0.95;1) = 3.84$.
[b]Computed $\chi^2$ values for the numerical (sub-1) $6 \times 6$ conditional covariance matrix (Table 5) vs the theoretical (sub-2) $6 \times 6$ covariance matrix (Table 6) extracted from the unconditional $7 \times 7$ covariance matrix.
[c]Computed $\chi^2$ values for the theoretical (sub-1) $6 \times 6$ conditional covariance matrix (Table 6) extracted from the $7 \times 7$ unconditional covariance matrix vs the theoretical (sub-2) $6 \times 6$ conditional covariance matrix (Table 9) extracted from the unconditional $8 \times 8$ covariance matrix.
[d]Computed $\chi^2$ values for the theoretical (sub-1) $6 \times 6$ conditional covariance matrix (Table 9) extracted from the $8 \times 8$ unconditional covariance matrix vs the theoretical (sub-2) $6 \times 6$ conditional covariance matrix (Table 11) extracted from the unconditional $9 \times 9$ covariance matrix.
[e]Computed $\chi^2$ values for the numerical (sub-1) $6 \times 6$ conditional covariance matrix (Table 5) vs the theoretical (sub-2) $6 \times 6$ covariance matrix (Table 9) extracted from the unconditional $8 \times 8$ convariance matrix.
[f]Computed $\chi^2 > 3.84$, therefore significant difference at the 95% confidence level.

## Conclusion

A comprehensive study has been performed to develop a conditional (fixed-time) covariance matrix of the state vector errors from an unconditional (variable time) covariance matrix of both the state vector errors and time errors. Using multivariate normal distribution and polynomial regression theories, a linear first-order regressional relationship between the state vector and time is inadequate to represent the Keplerian motion of a vehicle. Bartlett's test for equality of variances demonstrates that a conditional $6 \times 6$ covariance matrix of the state vector errors derived from an unconditional $8 \times 8$ covariance matrix of the state and time errors provides the advantages of simplicity and the accuracy required for the Commercial Titan launch vehicle performance error analysis. The analysis also shows that the numerically derived conditional state vector error covariance matrix is correct.

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**Table 8 The 8×8 unconditional space-time covariance matrix ($S_* S_*'$)**

| | $x$ | $y$ | $z$ | $v_x$ | $v_y$ | $v_z$ | | $t$ | $t^2$ |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | 3.446319514E+08 | −4.397717398E+09 | 2.405500483E+09 | 5.546920320E+06 | 2.625494404E+05 | 2.601032708E+04 | $x$ | 2.113219046E+05 | −2.973733296E+06 |
| $y$ | −4.397717398E+09 | 5.715647311E+10 | −3.126089447E+10 | −7.206372321E+07 | −3.416400053E+05 | −3.344658255E+05 | $y$ | −2.760092044E+06 | 4.362429356E+07 |
| $z$ | 2.405500483E+09 | −3.126089447E+10 | 1.709771125E+10 | 3.941434888E+07 | 1.868309445E+06 | 1.830740327E+05 | $x$ | 1.509466315E+06 | −2.386511163E+07 |
| $v_x$ | 5.546920320E+06 | −7.206372321E+07 | 3.941434888E+07 | 9.086080903E+04 | 4.303675468E+03 | 4.238548559E+02 | $v_x$ | 3.478539661E+03 | −5.517938763E+04 |
| $v_y$ | 2.625494404E+05 | −3.416400053E+06 | 1.868309445E+06 | 4.303675468E+03 | 2.174439599E+02 | 1.255615857E+01 | $v_y$ | 1.686395724E+02 | −1.500376592E+03 |
| $v_z$ | 2.601032708E+04 | −3.344658255E+05 | 1.830740327E+05 | 4.238548559E+02 | 1.255615857E+01 | 6.137732991E+00 | $v_z$ | 1.406644184E+01 | −8.710173038E+02 |
| $t$ | 2.113219046E+05 | −2.760092044E+06 | 1.509466315E+06 | 3.478539661E+03 | 1.686395724E+02 | 1.406644184E+01 | $t$ | 1.382011111E+02 | −1.798015817E+03 |
| $t^2$ | −2.973733296E+06 | 4.362429356E+07 | −2.386511163E+07 | −5.517938763E+04 | −1.500376592E+03 | −8.710173038E+02 | $t^2$ | −1.798015817E+03 | 1.543582699E+05 |

**Table 9 Theoretical 6 × 6 conditional covariance matrix computed from the 8 × 8 unconditional covariance matrix**

| | $x$ | $y$ | $z$ | $v_x$ | $v_y$ | $v_z$ |
|---|---|---|---|---|---|---|
| $x$ | 2.143725476E + 07 | − 1.814578566E + 08 | 9.966990775E + 07 | 2.332277416E + 05 | 4.283965548E + 03 | 4.884760755E + 03 |
| $y$ | − 1.814578566E + 08 | 1.763809980E + 09 | − 9.672766485E + 08 | − 2.249033315E + 06 | − 7.427216461E + 04 | − 2.877678528E + 04 |
| $z$ | 9.966990775E + 07 | − 9.672766485E + 08 | 5.304759920E + 08 | 1.233489400E + 06 | 4.052717881E + 04 | 1.590198130E + 04 |
| $v_x$ | 2.332277416E + 05 | − 2.249033315E + 06 | 1.233489400E + 06 | 2.869132498E + 03 | 9.192387467E + 01 | 3.828251023E + 01 |
| $v_y$ | 4.283965548E + 03 | − 7.427216461E + 04 | 4.052717881E + 04 | 9.192387467E + 01 | 9.177314345E + 00 | − 2.230149104E + 00 |
| $v_z$ | 4.884760755E + 03 | − 2.877678528E + 04 | 1.590198130E + 04 | 3.828251023E + 01 | − 2.230149104E + 00 | 2.429608046E + 00 |

**Table 10 The 9×9 unconditional covariance matrix ($S_* S_*'$)**

| | $x$ | $y$ | $z$ | $v_x$ | $v_y$ | $v_z$ | | $t$ | $t^2$ | $t^3$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 3.446319514E+08 | −4.397717398E+09 | 2.405500483E+09 | 5.546920320E+06 | 2.625494404E+05 | 2.601032708E+04 | $x$ | 2.113219046E+05 | −2.973733296E+06 | 1.768913641E+08 |
| $y$ | −4.397717398E+09 | 5.715647311E+10 | −3.126089447E+10 | −7.206372321E+07 | −3.416400053E+06 | −3.344658255E+05 | $y$ | −2.760092044E+06 | 4.362429356E+07 | −2.467117521E+09 |
| $z$ | 2.405500483E+09 | −3.126089447E+10 | 1.709771125E+10 | 3.941434888E+07 | 1.868309445E+06 | 1.830740327E+05 | $z$ | 1.509466315E+06 | −2.386511163E+07 | 1.349268435E+09 |
| $v_x$ | 5.546920320E+06 | −7.206372321E+07 | 3.941434888E+07 | 9.086080903E+04 | 4.303675468E+03 | 4.238548559E+02 | $v_x$ | 3.478539661E+03 | −5.517938763E+04 | 3.111987928E+06 |
| $v_y$ | 2.625494404E+05 | −3.416400053E+06 | 1.868309445E+06 | 4.303675468E+03 | 2.174439599E+02 | 1.255615857E+01 | $v_y$ | 1.686395724E+02 | −1.500376592E+03 | 1.265037259E+05 |
| $v_z$ | 2.601032708E+04 | −3.344658255E+05 | 1.830740327E+05 | 4.238548559E+02 | 1.255615857E+01 | 6.137732991E+00 | $v_z$ | 1.406644184E+01 | −8.710173038E+02 | 2.600196613E+04 |
| $t$ | 2.113219046E+05 | −2.760092044E+06 | 1.509466315E+06 | 3.478539661E+03 | 1.686395724E+02 | 1.406644184E+01 | $t$ | 1.382011111E+02 | −1.798015817E+03 | 1.157687024E+05 |
| $t^2$ | −2.973733296E+06 | 4.362429356E+07 | −2.386511163E+07 | −5.517938763E+04 | −1.500376592E+03 | −8.710173038E+02 | $t^2$ | −1.798015817E+03 | 1.543582699E+05 | −4.501011332E+06 |
| $t^3$ | 1.768913641E+08 | −2.467117521E+09 | 1.349268435E+09 | 3.111987928E+06 | 1.265037259E+05 | 2.600196613E+04 | $t^3$ | 1.157687024E+05 | −4.501011332E+06 | 1.713455795E+08 |

**Table 11 Theoretical 6 × 6 conditional covariance matrix computed from the 9 × 9 unconditional covariance matrix**

| | $x$ | $y$ | $z$ | $v_x$ | $v_y$ | $v_z$ |
|---|---|---|---|---|---|---|
| $x$ | 1.793001873E + 07 | − 1.563142593E + 08 | 8.581280819E + 07 | 1.999787577E + 05 | 6.478479542E + 03 | 2.633070672E + 03 |
| $y$ | − 1.563142593E + 08 | 1.583553938E + 09 | − 8.679342239E + 08 | − 2.010669243E + 06 | − 9.000477446E + 04 | − 1.263427643E + 04 |
| $z$ | 8.581280819E + 07 | − 8.679342239E + 08 | 4.757265521E + 08 | 1.102122570E + 06 | 4.919770987E + 04 | 7.005547347E + 03 |
| $v_x$ | 1.999787577E + 05 | − 2.010669243E + 06 | 1.102122570E + 06 | 2.553928467E + 03 | 1.127281091E + 02 | 1.693624018E + 01 |
| $v_y$ | 6.478479542E + 03 | − 9.000477446E + 04 | 4.919770987E + 04 | 1.127281091E + 02 | 7.804184155E + 00 | − 8.212432420E − 01 |
| $v_z$ | 2.633070672E + 03 | − 1.263427643E + 04 | 7.005547347E + 03 | 1.693624018E + 01 | − 8.212432420E − 01 | 9.839944085E − 01 |